• Volume 7C, Number 3, December 1959

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BULLETIN OF THE RESEARCH COUNCIL OF ISRAEL

Section C
TECHNOLOGY

Bull. Res. Counc. of Israel. C. Techn.

Incorporating the Scientific Publications of the Technion—Israel Institute of Technology, Haifa

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THE ISRAEL SOCIETY FOR THEORETICAL AND APPLIED MECHANICS

Programme of the

NINTH SCIENTIFIC CONFERENCE

Technion–Israel Institute of Technology
Haifa
December 30, 1959

First Session	GENERAL MEETING	Chairman: PROF. M. REINER
09.30—10.00	1. President's Report	
	2. Secretary's and Treasurer's Reports	
	3. Elections	
Second Session	MAIN LECTURE	Chairman: PROF. M. REINER
10.00—11.00	Some recent progress in stress waves and scabbing of materials	Prof. Norman Davids
Third Session	FLUID MECHANICS	Chairman: DR. A. BETZER
11.15—12.45	1. Unsteady flow of heat in gases	Dr. M. Hanin
	2. The viscosity of air at high rates of shear	E. Bousso
	3. Wind flow over hills. Study of energy pattern factor	J. Frenkiel
Fourth Session	PROPERTIES OF MATERIALS	Chairman: DR. D. ABIR
14.30—15.30	1. A graphical procedure for the determination of the effect of residual stress on uniaxial fatigue limit	Dr. D. Rosenthal
	2. On the dual type of fracture in hardened cement mortars	Ori Ishai
Fifth Session	ELASTICITY AND DYNAMICS	Chairman: DR. D. ROSENTHAL
15.45—18.15	1. Limit design of a system of crossed beams	A. Zaslavsky
	2. Thermal buckling of solid wings	Dr. J. Singer
	3. On boundary conditions in the bending of thin elastic plates	Dr. A. Werfel
	4. A large deflection criterion for circular plates	Dr. Z. Karni
	5. Semi-spherical head grinding set	J. Boas Popper

SOME RECENT PROGRESS IN STRESS WAVES AND SCABBING IN MATERIALS

NORMAN DAVIDS*

Department of Engineering Mechanics, Pennsylvania State University, Penn., U.S.A.

ABSTRACT

This paper gives a review of some recent progress in the general field of stress-waves in materials, with particular emphasis on certain phases developed by the author and his students which have not yet appeared in publication. Enough previous and general literature as well as fundamental material is cited to provide background. Treated in detail are the subjects of scabbing, multiple scabbing, and some graphical solutions for plastic wave propagation in one-dimension.

INTRODUCTION — GENERAL NATURE OF SCABBING

By "scabbing" we refer to a certain type of damage produced in a solid material by a suddenly applied pressure of high intensity at its boundary, such as an explosion or by a projectile. The name refers to the pieces of various shapes and sizes called "scabs" which tend to fracture from the specimen. The name is used because the predominant shapes are usually flat disks. The word "spall" is also used with the same meaning. In this paper we shall discuss the known causes of this phenomenon, and some of the possible methods for its prevention, also some of the uses to which scabbing data can be put in improving our basic knowledge of dynamical properties of materials.

The general study of scabbing leads into an inquiry of the behaviour of stress waves in general, and the resulting fractures. These differ from conditions of static loading in the following respects:

a. If a stress wave (or pulse) is sufficiently rapid, cracks which start to form may not have time to grow before the pulse has passed on and removed the stress.

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b. With a pulse of short duration, only a small part of the specimen is under stress at any one time and conditions may occur in one part of the specimen independently of the rest of it.

c. The dynamic elastic behaviour of many solids is known to differ—in many cases considerably—from that observed statically. In fact, at the very high rates of loading associated with intense stress-pulses, materials which ordinarily are ductile may become brittle. Steel, for example, takes on some of the properties of glass!

d. Boundary effects occur, the most important being that compression waves become reflected as tensile waves at free surfaces of the material, usually at the opposite end from the point of impact.

These aspects of stress-wave propagation, namely crack initiation time, instantaneous stress distributions, dynamic elastic behaviour, and reflection effects, form in themselves very broad areas of study and have many applications in addition to scabbing.

SOME SIMPLE EXPERIMENTS IN SCABBING

Figure 1 shows a rod or plate subjected to a sudden impact at the right end, either by a bullet or by an explosive. To produce a scab requires the pulse to be very intense, and of very short duration. Thus, for an aluminum or steel plate, for example, a pressure of the order of 1 million pounds per square inch is required and lasting for only a few microseconds. Indeed, such a short duration is essential to scab formation, for reasons which we shall see later.

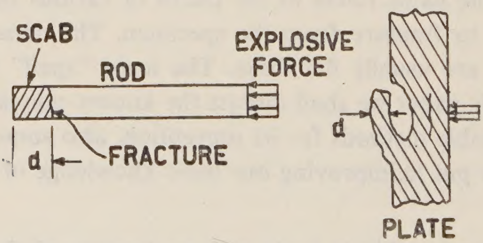

Figure 1
Impact and scabs on rod and plate

If the opposite end of the specimen is free, the formation of a scab will begin as a crack some distance back from this end. In the case of the rod a condition of plane waves may be assumed to exist so that the crack is practically simultaneous across the specimen. The separated piece to the left of this crack is then propelled to the left with considerable velocity determined by momentum balance.

In the case of the plate, where the situation is basically similar, a thin circular disk tends to tear itself out from the specimen. In many cases, where the process is only partially completed, the external appearance of the specimen shows a circular bulge of material across from the point of impact, and which may also show some partial cracks around the edges.

In both cases, where measurements have been taken, it is found that the thickness d is proportional to the time duration of the pulse. We can see at once that there is thus an upper limit to this time — namely when d reaches the entire length of the rod or plate thickness. In practice this limit is even smaller than that theoretically possible.

There are cases of scabbing in plate penetration problems which are due to shear-type failures rather than tensile separation as discussed above. These can usually be recognized by a highly burnished circular area over part of the specimen, indicative of slipping under high pressure and temperature. This will not be discussed further here.

STRESS WAVES

The basic idea of wave-propagation is the transmission of a wave-form in space or through a medium without essential loss of its shape. The quantity being varied could be referred to as a signal, excitation, input information, initial or boundary condition, depending on the type of problem. Complete preservation of shape is only possible in an ideal or dispersion-free medium. One usually classifies waves according to the type of quantity being varied. As examples, we might have stress-waves, displacement or velocity waves.

Acoustic waves in gases or liquids are transmissions of pressure through a medium. In solid media, because of the more complicated nature of the state of stress there, a richer variety of waves is possible. The main types are dilatation waves (of pressure or tension), distortion waves (of shear), and Rayleigh surface waves. The fact that any of these waves can be converted into another after reflection and the fact that they travel with different velocities, considerably complicates the analysis of these waves. Some velocity values for common materials are given by Kolsky[1], p. 201.

When seismic waves are propagated through the earth all the above types of waves are observed. In specimens such as armour plate the combined wave pattern is an important fact in determining the type and location of the failure.

Both the experimentalist interested in studying the effects of waves excited in specimens and the theoretical analyst concerned with solving initial and boundary-value problems have the choice basically of either pulse techniques or harmonic waves, although they may be limited by practical considerations. That the two approaches are theoretically equivalent is the content of the "Fourier Integral

Theorem", which states that any (non-periodic) wave form can be represented as the superposition of a set of harmonic waves, in which the frequency varies over the entire spectrum from zero to infinity. This property has basic importance not only in our subject but in optics, communication, and radar as well. When pulses are generated, the Fourier theorem enables us to correlate with the idea of frequency, by assigning to a pulse of time duration τ the frequency $1/\tau$. This, of course, is only an approximation, but it works well in most analyses.

It should be cautioned, however, that in practice, although it serves as a guide, it is not easy to apply this theorem to obtain numerical results, except approximately. Consequently, it is often a major problem to reconcile the results of pulse experiments with those obtained from continuous waves.

A considerable amount of data has now been assembled on the properties of materials in the range 20–100 kilocycles. For scabbing phenomena, however, the range of interest lies beyond, namely in the 1–10 megacycle range, leading to difficulties due to lack of knowledge of material properties, the fact that the wavelength becomes of the order of the size of the specimen, and that continuous waves cannot be generated with sufficient intensity. To minimize our ignorance regarding the first item, it is necessary to extrapolate the results known in the ultrasonic range. When the load fluctuates, as with vibrating bodies, fatigue and impact losses predominate. Under explosive or impact loading, where we have large forces acting for a very short time, we may have a material stress-strain curve of an entirely different character and one greatly sensitive to the rate of loading. What this will be depends of course on the material, there being two main categories. First are those which have a continuous stress-strain curve without any distinct flow at the yield point, e.g., aluminum or copper. Here the stress-strain curve tends to straighten up with higher strain rates. The second group consists of those materials which show a distinct flow at yielding as in the case of structural steel. Here more complicated delay-time effects occur (See also the articles of Hillier and Kolsky in Reference 2).

Early investigations in dynamic testing of materials were done by Karman[3], Taylor[4], and Rakhmatulin[5], who studied the strain wave propagated by longitudinal impact of rods. Clark and Duvez[6] pointed out the need for taking strain rate into account. Among others who have made mechanical property determinations at high rates of loading are Manjoine and Nadai[7], Kolsky[8], Ramberg and Irwin[9], and Campbell[10]. For an account of experimental methods and extensive bibliographies, see also the review articles by Davies[11,12].

MULTIPLE SCABBING

If the intensity of the explosive force at one end of a rod is strong enough, there may be up to as much as five fractured surfaces, as shown in Figure 2. It is useful to study since experimental arrangements utilizing them may be used to yield in-

formation and data about the actual shape of the pulses and dynamic fracture strength of the material. Details of the analysis of multiple scab formation are given by

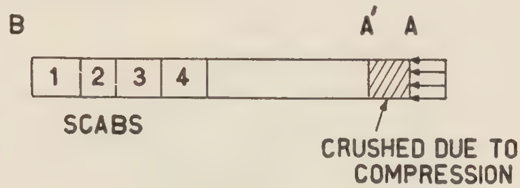

Figure 2

Multiple scabbing in a bar

Kumar and Davids[13]. The main factors governing the number and size of the scabs are:

(1) S_c, the dynamic strength of the material in compression,

(2) S_c/S_T, the ratio of the dynamic strength in compression and tension, and

(3) shape of the pulse.

For scabs to form at all, it is, of course, necessary that the peak pressure in the incident pulse be greater than S_T, and that S_c/S_T should be larger than unity.

Kumar gave a graphical approach to scab length determination in bars if the shape of the pulse is known. An example of this, for a case where the peak pressure exceeds the compressive strength S_c of the material, is shown in Figure 3. Here a part of the bar is crushed initially until the head of the pressure pulse is cut off (dotted part in Figure 3). The actual pulse is thus reduced to the effective pulse which propagates further into the bar causing scabbing. In order to determine the length of various scabs, we plot the given pressure pulse as pressure P vs distance x and mark off intercepts of length S_T starting from the top of the peak pressure P_m in Figure 3. Differences of the abscissae of the various intercepted points give the lengths of the scabs. This procedure requires the assumptions that (1) peak pressure is attained instantaneously, (2) after peak the pressure falls slowly and monotonically. These conditions are usually satisfied in a pressure pulse generated by a single explosion.

An inverse procedure was also given by Kumar, whereby the pulse shape can be determined from the lengths of a group of scabs in a specimen. For example, in

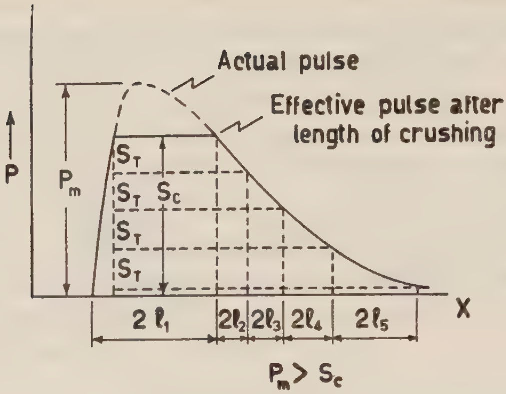

Figure 3

Graphical construction of scab length from given pulse

Figure 4, if l_1, l_2, l_3 . . . etc. represent the various lengths of the first, second, third, . . . scabs, a number of points on the pulse-shape curve is determined equal to the number of scabs. This is done by first plotting point A having the ordinate S_T, then plotting the next point B with ordinate $2S_T$, at a distance $2l_5$ to the left of A. The successive points C, D, . . . are determined by marking off successively to the left $2l_4$, $2l_3$, . . . and erecting ordinates $3S_T$, . . . and so on. Passing a smooth curve through these points gives a set of points of the after-part of the pulse. The steeply rising part is taken as vertical.

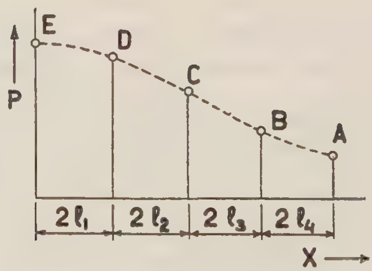

Figure 4

Inverse construction for pulses

In order to obtain the greatest number of scabs, a material is desirable which satisfies the following conditions: (1) S_c/S_T should be as high as possible, and (2), the stress-strain diagram should be as close to linearity as possible up to fracture. It is not easy to find materials which satisfy this unusual combination of requirements.

An example of one that does is Plaster of Paris. This material has a stress-strain relationship which is practically linear, both in tension and in compression, up to failure provided constant humidity conditions are maintained. Also the ratio of S_c to S_T is approximately 4, so that for a strong enough pulse, about 4 scabs might be expected. In addition specimens of this material are easy to make.

MULTIPLE SCABBING EXPERIMENTS

As suggested by Kumar, experiments were carried out by A. Jones[14] on multiple scabbing in Plaster of Paris. Up to 5 scabs were produced by a pellet-impact arrangement, a schematic diagram of which is shown in Figure 5. Air pressure at 30 to 40 pounds per square inch was used to propel the pellet at velocities of 50–60 feet per second. The holes near the muzzle of the tube were for pressure relief after impact.

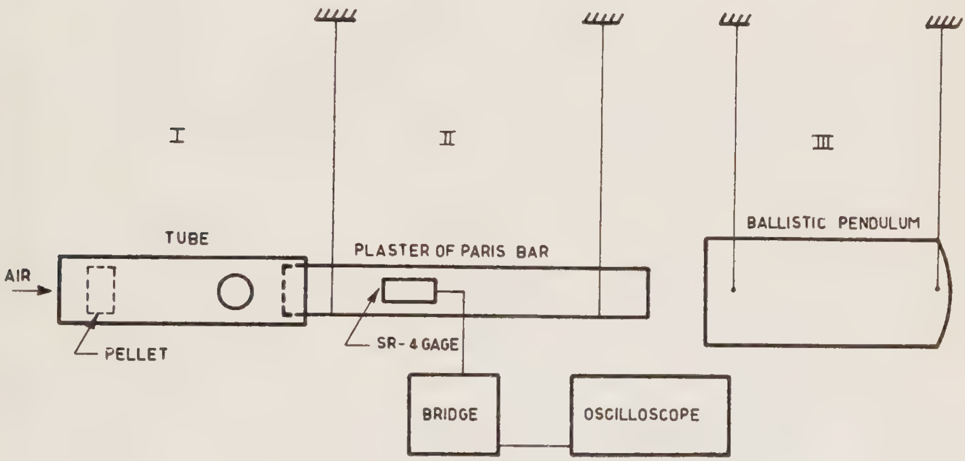

Figure 5

Schematic experimental arrangement for scabbing in Plaster of Paris bars

The induced pulse intensity was picked up by means of an SR-4 strain gauge attached to the bar. The momentum trapped in the flying scabs was measured by a ballistic pendulum. Figure 6 shows two bars, scabbed in this way. It can be seen that, while the scabs form in the region expected, the specific location of each fracture is irregular. Jones found this to be due primarily to the fact that the pellet arrangement gave too long a rise time to the pulse. This, as well as the time delay for microcracks to develop into a full-fractured surface, as noted by Broberg[15], influences the location of fractures and makes it somewhat inexact.

In order to show that Plaster of Paris may be used as model for more complicated specimens, Jones also conducted a few tests on plates. By detonating 0.4 g lead azide spread in a thin layer over a circle one inch in diameter on the upper surface of a

Figure 6

Photograph of multiple scabbing in bars

one-inch thick plate, he obtained a clearly defined scab at the opposite face. The scab had an internal crack parallel to the free surface, and therefore was probably the beginning of another scab.

PLASTIC WAVES

If the outcome of any wave-shape stressed in the material were known, its dynamical properties could be inferred and also vice-versa. In practice we have only an imperfect knowledge of either of these sets of quantities, and considerable analytical skill is necessary on even the simplest of problems. Most of the work in pulse propagation discussed above was done in the elastic range. Since the intensity of loading is high-enough in many impacts to exceed this range, the plastic behaviour of the material needs to be taken into account in the problem.

Because of the complexities in the problem, most of the past work in this field has been restricted to one-dimensional propagation in thin bars. The propagation of plastic stresses was first studied by Donnell[16] in 1930. The subject was not given much further attention until World War II, when an extensive study was made of plastic waves due to a longitudinal impact on a long, thin wire independently in the U.S. by von Karman and Duwez[3], in U.K. by Taylor[4], and in U.S.S.R. by Rakhmatulin[5]. Further work, both experimental and theoretical, was carried out by White and Griffis[17,18], Clark and Duwez[6], DeJuhasz[19], Lee and Tupper[20], Malvern[21], Kolsky[22] and Rinehart and Pearson[23], in whose book extensive bibliographies are given.

On the theoretical side, graphical or semi-analytical methods are prevalent. One of the assumptions often made is that the pulse shape being propagated is rectangular. Since this is only justified in some restricted cases, Davids and Kumar[24] extended a graphical procedure of DeJuhasz[19] useful for general pulse shapes in one-dimensional

specimens including the triangular type occurring in scabbing fracture problems. Such procedures help to obtain solutions to many propagation problems too complicated to solve by other methods, anf may be of advantage in giving direct physical insight into the manner of formation or interaction of waves.

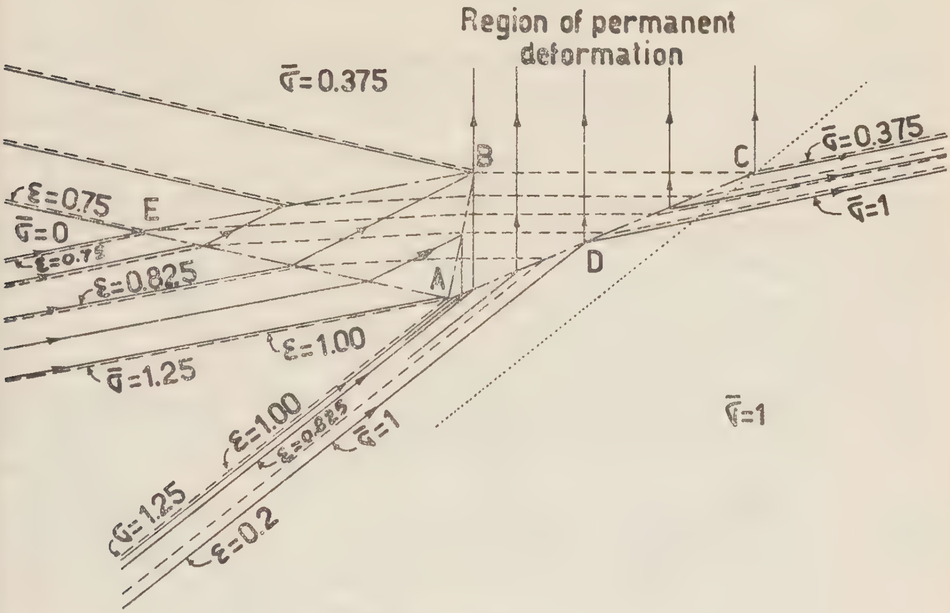

Figure 7
Schematic overtaking of plastic wave by unloading elastic wave

By plotting a set of contours of strain $\varepsilon(x, t)$ and particle velocity $v(x, t)$ the geometric condition of mutual intersection is that the product of slopes be ρ/S, where ρ is the density and S the tangent modulus of the material. With this and other basic relations formulated as "rules" in reference 24 one may draw contour diagrams for many types of initial conditions and stress-strain behaviour, including that of residual strain upon unloading. A single example must suffice here. Figure 7 shows the interaction region for the case of a plastic wave being overtaken by a faster moving elastic wave, generated by elastic unloading of a material with the stress-strain characteristic shown. Such an unloading wave tends to reflect off the plastic one, thereby weakening it. If the elastic wave is sufficiently strong, it may absorb the plastic wave completely, as in the figure. One feature in all such cases is that a permanently-strained region develops in the material at the place of interaction.

SHOCK FRONT FORMATION

Brief mention will be made of a study on the formation of shock fronts under nonlinear stress-strain behaviour of the material made by Mr. Shoi-Yean Hwang[25].

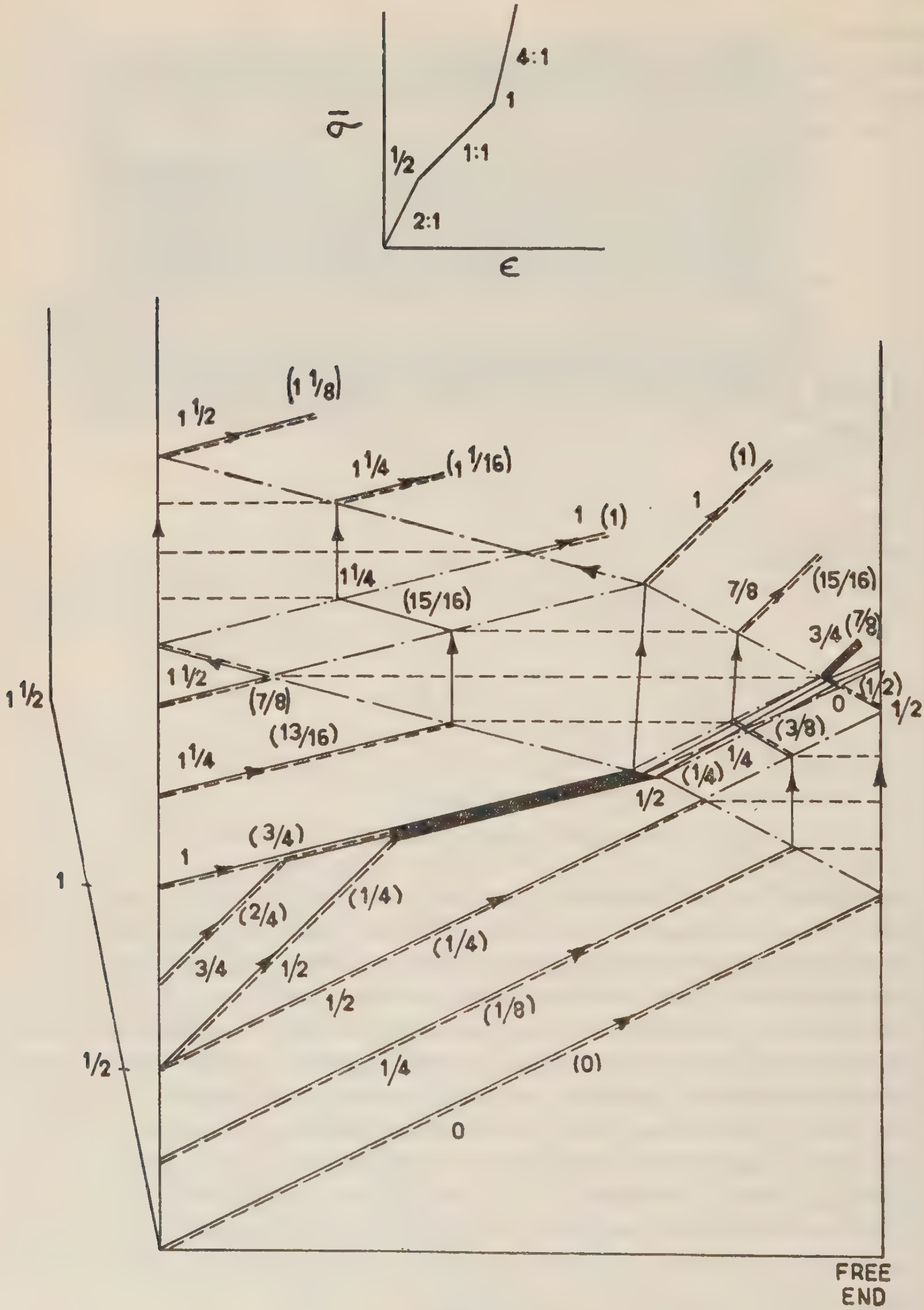

Figure 8
Interference of three waves, forming a shock front (light solid line = stress contour, dotted line = velocity, heavy line = shock front. Unenclosed numbers represent non-dimensionalized stress values on the contours, numbers in parenthesis, velocity.)

This is a continuation of the work by Kumar and Davids[26], and also by Lee[27] and others. When such non-linear behaviour is present, the phenomena which result may be classified according as the wave velocity under high loading is smaller or larger than under low loading. In the former case we have the plastic wave described by Lee and others, but in the latter, shock fronts tend to be formed analogous to those in fluids.

When the stress-strain diagram of the material has a portion where the slope is increasing as the strain increases, such shock fronts tend to appear. Examples of material in this category are types of raw rubber, soft rubber and foam plastics. Further complications in the waves propagated in such materials arise from repeated loading cycles, resulting in residual strains. An example of the resulting diagrams is given in Figure 8. This shows what might happen theoretically in a material where there is a middle range of stress with a lower wave velocity than in the low or high range of stress, as shown in the σ,ε-curve. A soft rubber might exhibit such a behaviour. In the main figure the left end of a bar is loaded uniformly to a non-dimensionalized stress value of $1\frac{1}{2}$, then remains constant. The faster wave from $\sigma = \frac{1}{2}$ to $\sigma = 1$ overtakes the latter wave and develops the shock front, shown dark, which disappears before reaching the free right end of the bar because of interaction with the reflected wave.

Further cases are discussed in the cited reference, including some where a region of permanent deformation is generated by the two waves, and one where scabbing results in the material through such interaction.

THE GENERAL IMPACT PROBLEM

The work we have been describing is only a small part of the general problem — what happens when two solids interact — generally at high speeds. The study of penetration of a small body into a larger one is an important practical aspect of this. The literature, although large, is not of the same order as in many other branches of mechanics. We shall close this account with a brief review and classification of some of the recent literature. The subject started essentially with Hertz[28], who extended his theory of contact to impacting solids. In recent times the general problem was attacked from the mathematical, physical, metallurgical, and experimental point of view. The French made many basic studies after the first World War[29]. Broad, general treatment and more complete reviews are given by Davies[12] and Rinehart[23]. The basic mathematical theory of stress waves in general is given by Kolsky[1].

a. Mathematical

Craggs[30] has analyzed mathematically the effect of speed and cone angle of a projectile on bulging of plates, generalizing a procedure similar to that used by G. I.

Taylor for a wire. Eringen[31] analyzed flexural deflections for various edge conditions, with examples. Further analysis in plates using linear theory were made by Kane and Mindlin[32], Gazis and Mindlin[33], Greenspon[34], Davids and Kumar[35,36], Goodman[37], Thiruvenkatachar[38], and extended in Russia to include plasticity by Cristescu[39].

Pekeris[40] developed solutions to certain geophysical layer problems, whose procedures are also applicable to plate mechanics. His article in reference 2 describes recent results obtained by use of the WEIZAC Computer. Mencher[41] also solved a layer problem by Caignard's method, to be discussed further below.

Equations of deformation of simply-supported and of built-in plastic plates exposed to a strong blast were given by Wang[42,43].

Wood[44] advanced substantially the theory of the combined effects of elastic and plastic wave propagation. Pursey[45] approached the propagation of elastic waves in plates from the point of view of integral transformations.

Thomas[46] approached the problem of weak shock propagation by setting up the equation of state and discontinuity conditions for a solid.

Lee[47,48] extended Karman's theory and by a method based on the concept of characteristics, solved some important examples.

The expansion method suggested by Caignard[49] for solving geophysical layer problems has been used by Davids[50] for obtaining the axial solution for stresses through a plate under exponentially-declining type impact at a point. This method obviates the necessity of having to work out the double contour integrals appearing in the formal solutions for arbitrary time and spatial variation. Instead, the results appear in the physically-natural form of a series of terms representing successively reflected waves. In a plate such as steel or aluminum, say, the waves reduce in amplitude rapidly enough so that only the first few reflections need be considered. Figure 9, taken from reference[50] shows how the incident wave behaves as it passes axially through the plate. We note the presence of a stressed zone, of steadily rising intensity, between the initial front moving with the dilatation velocity and the negative front arriving with the distortion velocity. The peak intensity is reached just ahead of this. There is also a δ-function singularity at the first front, studied by Broberg[51] which falls as $1/R$.

Impacts with prescribed displacement boundary conditions, or for shear stress inputs, have been analyzed by Pytel[52]. Calculations for the displacements at the point opposite that of impact on a plate, together with measurements, were given by Broberg[51], and by Allen[53].

An analysis is still in progress by the author on certain questions connected with the behaviour of a wave in the vicinity of its source, supposed as a concentrated normal load applied as a step-function. For we must note that, in order to avoid the difficulty of an infinite stress and an infinite displacement at the load, it is neces-

sary, as in the static problem, to cut out a hemispherical element there. The lack of static equilibrium requires examining the legitimacy of this approach. By means of a certain spherical wave expansion it can be shown that there is a set of stress waves which approach the static loading conditions as the radius $R \to 0$. Fuller details are to be published.

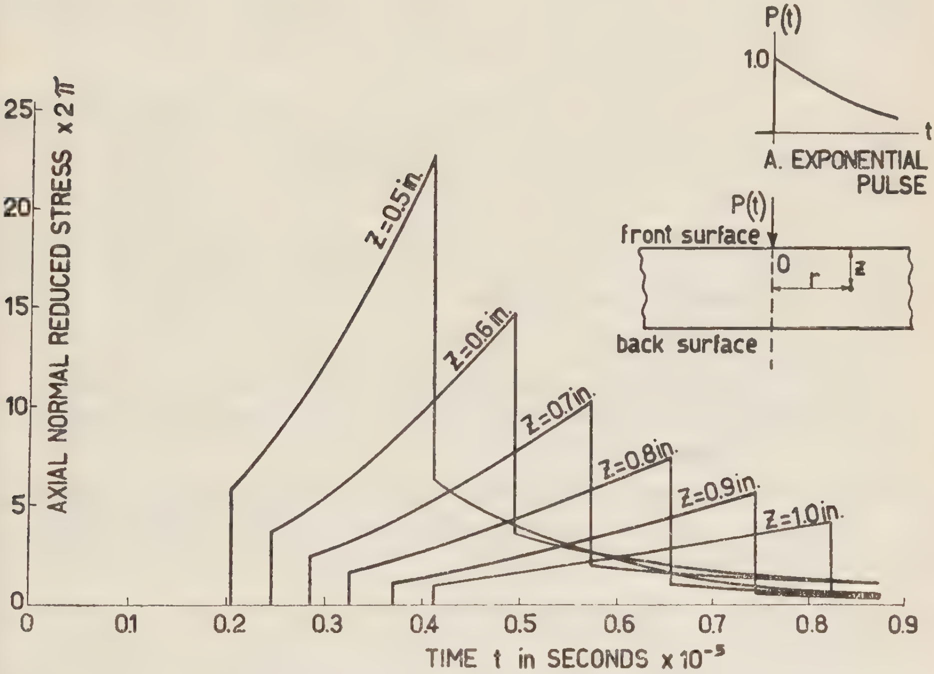

Figure 9

Axial stress amplitudes of incident waves for exponential pulse

b. Experimental

Van Valkenburg, Clay and Huth[54], who were interested in the penetration effects of ultra-high velocities (meteors on the metallic skins of satellites), made ballistic measurements at 1 to 5 mm/μ sec. They also treated the penetration mechanics of the problem.

Much useful data on steel-fragment velocities after perforation were obtained by Spells[55].

In Japan, Nishiwaki[56] was able to develop a method for measuring the velocity drop of a penetrating bullet, as well as a theoretical formula.

Masket[57] perfected an optical chronograph method at the Naval Research Laboratory by which considerable additional data were obtained on these decelerations, as well as their strain-rate and inertia effects.

Allen and McCray[58] made measurements of surface particle velocities and found elastic and plastic waves in thick steel plates under explosive attack as well as permanent strain configurations.

Evans and Taylor[59] made studies of explosives detonated in contact with steel plates, in addition to determining mathematically the maximum stress taking into account shock and pulse reflections.

Evidence for shock waves were obtained by Feder, Gibbons, Gilbert and Offenbacher[60] in plastics and by Savitt[61] in naval brass plates.

The surface motion of a plate under explosive attack was studied photographically by Shreffler[62]. By similar methods, Vigness[63] made some very basic strain and displacement measurements of transverse waves in circular plates under velocity impact.

Experiments in simulated media, such as Plaster of Paris, were made by Rinehart and White[64], and in plastics by Van Valkenburg[65].

Gerard and Papirno[66] are in process of making high-speed strain measurements to further test the von Karman theory of plastic wave propagation. Stresses in projectiles were also measured by Bluhm[67].

Bell[68] has made a detailed experimental study of large-amplitude plastic strain waves in aluminum, using a technique of ruling a diffraction grating, developed by him, on the specimen.

c. Energy

An analysis was made by Miller and Pursey[69] on energy partitions among the different waves in a semi-infinite medium.

The energy requirements for exciting free extensional vibrations in finite plates were analyzed by Foux and Davids[70]. The relative energy level for third mode was found to be lower than for second mode.

FURTHER WORK

Much more work needs to be done on the relationship between scabbing and the mechanical properties of materials. With improvement in techniques it could be used as a tool in studying material behaviour under very high rates of strain. Also an effort is needed to systematize the extensive amount of general data now available. Improved mathematical analyses are still needed, especially close up towards sources of disturbances, where "long-wave" approximations cannot be used.

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UNSTEADY FLOW OF HEAT IN GASES

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SUMMARY

A theoretical study is made of the changes of state and the motions which are produced in a gas, originally resting in a uniform state, when the tempreature of its plane boundary is suddenly raised or lowered. Because of the compressibility, the heat conducted through the gas produces not only temperature variations but also variations of the pressure and the density, which in turn induce a motion of the gas in the direction normal to the boundary. Description and evaluation of these phenomena are obtained by solving the Navier-Stokes equations for a heat-conducting, viscous, compressible fluid. To facilitate the solution it is assumed that the relative temperature change at the boundary is not large; this assumption permits to linearize the equations, which are then solved by applying the Laplace transformation. Interpretation of the transformed solution yields (1) the variation with time of the temperature, pressure, density and heat-transfer rate at the boundary, (2) an approximate solution giving the changes of state and the motion of the gas at small values of time, (3) an asymptotic solution valid for large times. These results show that in the initial stage large temperature and pressure gradients appear near the boundary, while the induced velocity and the variation of density are still small. As the time grows, the variations of temperature and pressure spread into the gas, the pressure excess at the boundary decreases, the normal velocity and the variations of density become appreciable. At large times the pattern of the flow appears as a superposition of two distinct components. One of these components involves changes of temperture and pressure only, the temperature variation being the same as for an incompressible medium. The other component is a wave-like disturbance which propagates into the gas with the speed of sound and diffuses itself gradually through the action of viscosity.

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THE VISCOSITY OF AIR AT HIGH RATES OF SHEAR

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ABSTRACT

An instrument for the measurement of the viscosity of air at high rates of shear is described together with the reasons for the design chosen. The values obtained for the viscosity were lower than those measured by other methods. An explanation is offered on the basis of slippage of gas molecules adjacent to the surfaces.

Experiments with Prof. Reiner's centripetal air pump suggested to me an instrument for measuring the viscosity of air at high rates of shear.

In this instrument, a flat rotor (R) is rotated against another flat surface (S). It has been found that in this case, when the gap between the rotor and stator is small enough, air is drawn in centripetally, and can balance, by means of the internal pressure produced, external forces which tend to bring the plates together.

The order of this air gap is of microns (1 to 10μ). When the two surfaces move

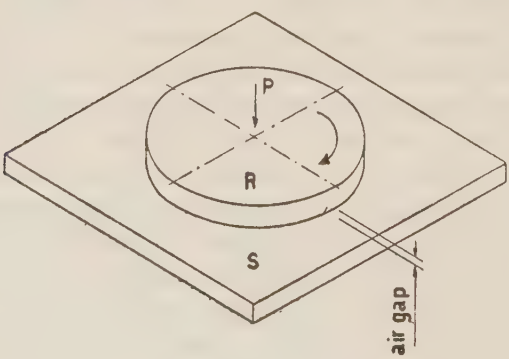

Figure 1
Schematic description of Reiner effect

relative to one another, the velocity gradient γ is proportional to the velocity and inversely proportional to the distance between them. In the case of the centripetal pump a high velocity gradient is obtained even with relatively small velocities because of the extremely small gaps.

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Velocity gradients of the order of $10^7 sec^{-1}$ and higher are obtained without difficulty. This gradient is equivalent to that obtained when a surface moves at a velocity of 10 km/sec relative to another at a distance of 1 mm.

This is a favourable factor from the point of view of the forces to be measured. Because of the low value of the viscosity of air it is difficult to obtain forces or moments large enough to be compared with Coulomb friction forces.

Various designs embodying one rotor turning against a pivoted stator were discarded because, in general, with these designs the relative weight (and consequently the Coulomb friction force) of the measuring element were too large by comparison with estimated forces due to the viscosity of the air.

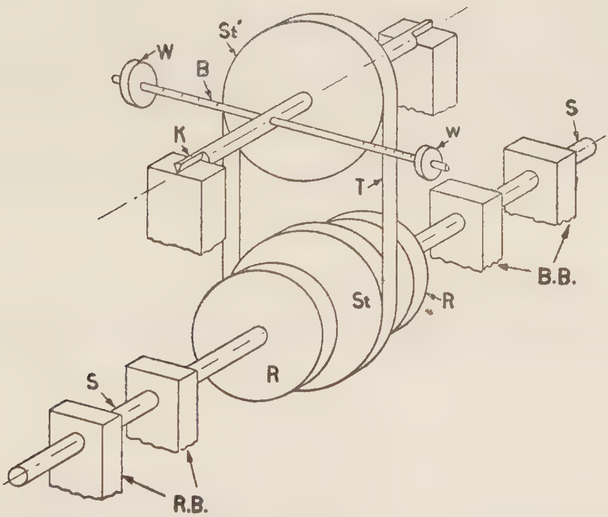

Figure 2
Schematic description of viscometer

In the design adopted two rotors (R, R) mounted on collinear shafts (S, S) are rotated against a fixed stator (St). The stator is suspended by means of a band of extra-fine recording tape (T) (thickness 25μ) onto another plate (St′) of equal diameter mounted on knife edges (K). The rotors are mounted on spehrical seats which permit them to align themselves dynamically perpendicular to the axis of rotation. One shaft is mounted on spherical roller bearings B. B. while the other is mounted on cylindrical roller bearings R. B. This permits the air gap to adjust itself to the external axial force which acts on the shaft mounted on the cylindrical roller bearings. This force is produced by a cylindrical compression spring, and can be varied by compressing it. The stator plate is slightly larger that the rotors (stator 64 mm diam., rotor 60 mm diam.). This assures complete coverage of the rotor surface by the stator at

all times. The viscous moment produced by the two rotating rotors is transmitted by means of the tape to the upper disc where it is measured by means of sliding weights (WW) mounted on a beam (B) rigidly attached to the upper plate. Its value is measured directly in gram-millimetres.

The friction in this instrument is derived from two sources: rolling friction at the knife-edges and hysteresis of the tape material. Both of these are extremely low, being less than 1 gram-millimetre (between 0.1 to 0.2% of the moment measured), and provide the smallest source of error in the measurements.

The distance between the plates is measured by means of a capacity bridge, the rotors and stator forming the plates of the condenser. The conducting suspension tape provides a simple solution to the problem of insulating the stator and rotors. The speed (in r.p.m.) of the rotors is measured by means of a discharge tube stroboscope. (The accuracy of the capacity bridge is $\pm 2\%$, and that of the stroboscope $\pm \frac{1}{2}\%$). The viscosity of air can be obtained from the formula $M = (\pi\omega\eta/2t)\, r^4$, where M is the moment for one plate in mm-kg, ω the speed of rotation in rad/sec, t the air gap in mm, r the rotor radius in mm, η the viscosity in kg sec/mm^2. For two plates we obtain for the viscosity

$$\eta = \frac{Mt}{\pi\omega} \cdot \frac{1}{r^4}$$

If we write $\omega = 2\pi n$ and $A = (K_1/\mathrm{C})$, where n is the speed in r.p.m. and c the capacitance in $m\mu F$, we obtain a formula for η in the directly measured variables

$$\eta = K \frac{M}{n\mathrm{C}}$$

It is obvious that when working with gaps of the order of microns, exceedingly flat surfaces are required as well as accurately dimensioned rotors. In this case the surfaces were hand-ground, lapped and polished to 0.1μ flatness and surface finish.

A large number of readings was taken at various speeds and with different axial forces. In general the range of the readings was within $\pm 3\%$ of the means. The means of the various series were about 5% lower than the value for the viscosity of air at the given temperature as recorded in tables. In order to understand this phenomenon it is necessary to refer to the kinetic theory of gases.

When speaking of the viscosity of a gas in a broad sense, it is generally assumed that the gas is a continuous medium, and that at the boundaries the velocity of the gas is zero. Actually the gas is composed of molecules moving in all directions at high velocities. At each temperature there is a mean molecular velocity which varies roughly as the square root of the absolute temperature. Because of the high velocity and density, frequent collisions occur between the molecules themselves and between them and the walls of the container. The mean free path in air at S.T.P. is of the

order of 0.1μ. When the mean free path approaches the order of the dimensions of the container (for example, by greatly reducing the pressure of the gas or the dimensions of the container) the assumption of a continuous medium and zero velocity at the boundary relative to it is no longer justifiable. In experiments at low pressures in capillary tubes it has been found that the rate of flow exceeds that to be expected from the Poiseuille formula.

Knudsen and Timiriazeff [2] have found that in order to allow for slippage at the wall the viscosity formula must be written in the form

$$\eta = \left(\frac{M}{v}\right)(d + 2e),$$

where M is the momentum transmitted in umit time, v the speed, d the distance between the two surfaces and e a value of the order of the mean free path of the gas (between 0.7 and 0.9 of it). With this correction, the values of the viscosities obtained are in excellent agreement with those derived by other methods.

An interesting consequence of this effect is that the viscous resistance does not increase inversely with the distance but tends to a definite limit, below which a decrease in distance would cause no further increase in the viscous resistance at a given speed.

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WIND FLOW OVER HILLS
STUDY OF ENERGY PATTERN FACTOR

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SUMMARY

One of the aims of the current investigation of wind conditions at two hilly sites, which have been selected on the basis of the general wind survey as suitable for the erection of experimental large-scale wind power plants, is to study the question of the energy pattern factor.

This question arises while evaluating the total energy available in the wind, As is known, the power in the wind is proprtional to the cube of the wind velocity

$$P = kv^3$$

the constant k depending on the air density, the choice of units for power wind velocity, and the area through which the wind passes normally. The energy in the wind is then given by

$$E = \int p dt = k \int v^3 dt$$

In order to evaluate this energy over a period of time (say a year) one uses an observed frequency distribution of mean wind speeds over fixed (say, hourly) intervals. In this way one obtains

$$E^* = k \sum_n \bar{v}_n^3 \ \ \Delta t_n$$

($\bar{v}_n$ — mean wind velocity over the interval of time Δt_n)

But the wind does not blow steadily throughout any interval of time Δt_n, and therefore

$$E^* < E$$

The energy pattern factor K_e is then defined as

$$K_e = E/E^*$$

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By choosing smaller and smaller intervals of time Δt, we shall obtain a series of E^* approaching E in the limit. For practical purposes we are investigating the following three sequences of time intervals:

1) 24 hours — 1 hour — 10 minutes
2) 1 hour — 10 minutes — 1 minute
3) 10 minutes — 1 minute — 2 seconds

using a different type of anemometer for every sequence. Obviously, the amount of energy calculated on the basis of 2-second intervals is still less than the true energy in the wind. However, it is an amply sufficient approximation for wind power utilization purposes, and the knowledge of

$$K_e^* = \frac{E^*_{\Delta t = 2\mathrm{sec}}}{E^*_{\Delta t = 1\mathrm{hr}}}$$

is of considerable importance for the evaluation of the accurate amount of utilizable energy from the wind.

A GRAPHICAL PROCEDURE FOR THE DETERMINATION OF THE EFFECT OF RESIDUAL STRESS ON UNIAXIAL FATIGUE LIMIT

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ABSTRACT

A simple graphical procedure based on previous findings, is described to compute the effect of residual stress on uniaxial fatigue limit. Various applications are discussed and some limitations are pointed out.

NOTATION

S_1 — applied uniaxial fatigue limit
R_1 — residual principal stress in the direction of S_1
R_2 — residual principal stress perpendicular to R_1
YS — yield stress
RF — reversed fatigue limit (for zero mean stress)
EL — endurance limit (reversed fatigue limit for notch free specimen)
ΔR_1, ΔR_2 — relief (reduction) of residual stress by applied loading
α — material constant (a fraction)
f_1 — S_1/RF
s_1 — R_1/RF, s_2 — R_2/RF
k — YS/RF
k_o — YS/EL
K_t — stress concentration factor
K_f — stress reduction factor (also called fatigue notch factor**)

THE GOVERNING RELATIONS

The effect of residual stress on fatigue and the concomitant effect of applied load on the residual stress have been analyzed previously***. It has been found that these effects could be reasonably well predicted using relations as follows:

1. The maximum shear stress criterion for yielding;
2. the experimentally established linear relation between the mean (static) and the alternating fatigue stress;
3. the experimentally established relationship between the amounts of residual stress relieved by applied loading in two mutually perpendicular directions.

Since surface fatigue is by far the more frequent cause of failure we shall limit our considerations to the biaxial residual state of stress which prevails on the surface.

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** For details, see *Manual on Fatigue Testing*, Am. Soc. for Testing Materials, Philadelphia, 1949, p. 3,

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We shall further limit the discussion to the case where the residual stresses on the surface are of the same sign, either both tensile or both compressive. One might conceivably produce residual stresses of opposite sign, yet all practical methods employed in industry are known to set up stresses only of the same sign. We can finally assume without great loss of generality that the applied stress is of a purely alternating nature, i.e. of equal positive and negative amplitude.

With these limitations, using the notation given in the beginning of this note we have the following expressions for the three relationships mentioned above:

$$|S_1| + |R_1| \leqslant YS \tag{1}$$

$$|S_1| = RF - \alpha(R_1 + R_2) \tag{2}$$

$$\Delta R_2 = \tfrac{1}{2}\Delta R_1 \tag{3}$$

Using the dimensionless numbers given in the notation we transform the above relations as follows:

$$\frac{|f_1|}{\kappa} + |s_1| \leqslant 1 \tag{4}$$

$$|f_1| = 1 - \alpha k(s_1 + s_2) \tag{5}$$

$$\Delta s_2 = \tfrac{1}{2}\Delta s_1 \tag{6}$$

In these equations α is a material constant. For notch free specimens $k = k_0$, the ratio of the YS to the endurance limit EL. In the presence of notches $k = k_0\, K_f/K_t$, where K_t — the stress concentration factor — depends on the notch geometry, and K_f — the strength reduction factor — on both notch geometry and the material.

Having s_1 and s_2, the value of f_1, the fatigue limit in the presence of residual stress, can be found as a multiple of RF, the fatigue limit in the absence of residual stress, using equation (5). However, if relation (4) is not satisfied, a process of iteration is necessary which also involves equation (6).

The purpose of this note is to present a graphical procedure which facilitates the iterative process considerably and which in addition permits to evaluate the role of various factors in a more straightforward fashion than the direct analysis of equations (4) to (6).

GRAPHICAL PROCEDURE

We begin by eliminating k from equations (4) and (5), which leads to the relation

$$\frac{1 - |f_1|}{|f_1|} = \alpha \frac{s_1 + s_2}{1 - |s_1|} \tag{7}$$

This relation is represented in Figure 1 for the case of aluminum alloys for which α is found to be 0.25. The quantities $|f_1|$ and s_1 are plotted as ordinates and abscissae,

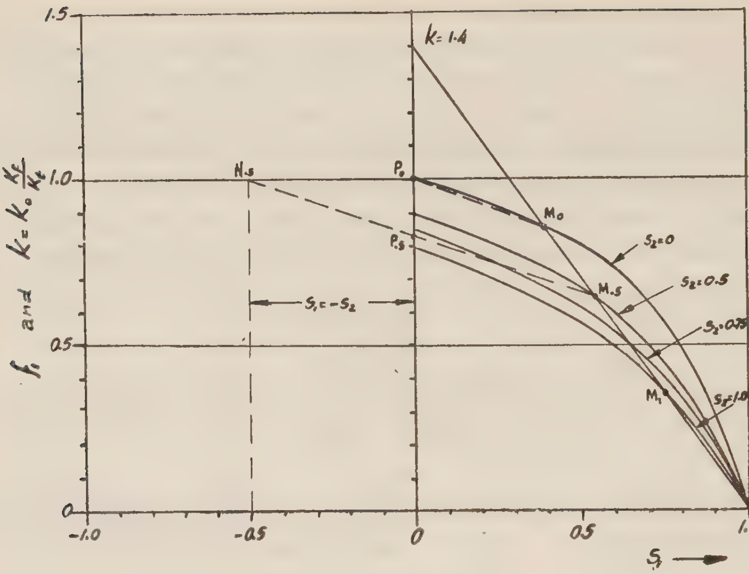

Figure 1

Influence of s_1 on fatigue limit f_1 P_0M_0 for $s_2=0$ $P_{0\cdot5}M_{0\cdot5}$ for $s_2=0.5$

respectively, and s_2 is used as a parameter. It will be noted that the values of k are also plotted on the vertical axis of ordinates. If a straight line is passed between a point on the vertical axis corresponding to a given value of k (e.g. $k = 1.4$) and the point on the horizontal axis corresponding to $s_1 = 1$, all points on this line, as is easily seen, satisfy relation (4) with the sign of equality. In addition, the points of intersection, M_0, $M_{0\cdot5}$,..., M_1, with curves $s_2 = 0$, $s_2 = 0.5$... $s_2 = 1$ also satisfy equation (5). The point N for which $s_1 = -s_2$ is another point belonging to (5). Hence line PM* represents the relation (5). Point M corresponds to the maximum amount of residual stress s_1 that can remain on application of the safe fatigue stress $|f_1|$. The construction is valid only for positive values of s_1 and s_2. However, the effect of compressive residual stress can be easily evaluated by adding to, instead of subtracting from unity the value $1 - |f_1|$.

DISCUSSION

1. The iterative procedure for values of s_1 exceeding those satisfying relation (4) is illustrated in Figure 2. Suppose e.g. $k = 1.4$, $s_1 = 1$, $s_2 = 0.75$. The point of intersection $M_{\cdot75}$ satisfying (4) and (7) gives $s_1 = 0.64$, instead of 1. Hence there is a relief Δs_1 of $s_1 = 0.36$. But according to (6) this relief entails a corresponding relief $\Delta s_2 = 0.18$ which causes the point of intersection to shift upward to curve $s_2 = 0.57$.

* In this case $M_{0\cdot5}$ $P_{0\cdot5}$

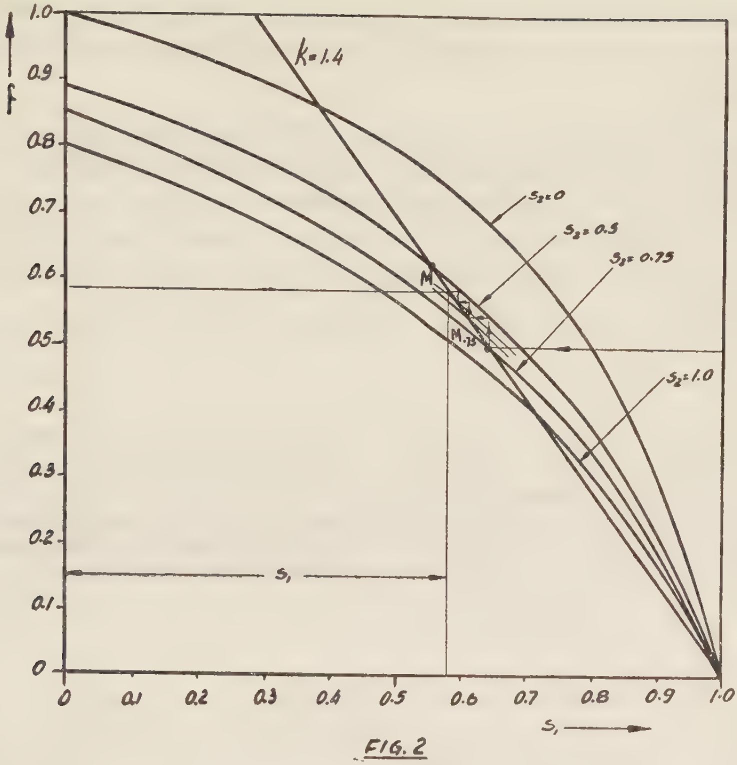

Figure 2

Iterative procedure for determining f with corresponding relief of s_1 and s_2

An additional relief of s_1 follows which is equal to 0.03. The corresponding relief of s_2 will bring the new point of intersection to 0.56. A third relief will give $s_2 = 0.55$. The next correction will be so small that it can be neglected. The final values therefore are:

$$f_1 = 0.58 \quad s_1 = 0.58 \quad s_2 = 0.55$$

2. The graph reveals that the effect of a biaxial residual stress on fatigue is much greater than that of a uniaxial one. The influence of s_2 is particularly significant. Since the direction of the applied stress f_1 is generally known, considerable improvement of fatigue can be obtained by inducing large residual compression perpendicularly to f_1.

3. Contrary to common belief the presence of a notch is apt to reduce rather than enhance the effect of residual stress on fatigue. This follows from the fact that K_f/K_t appearing in the expression of k is generally a fraction lying between 0.5 and 1. This circumstance makes it highly unlikely that uniaxial fatigue limit

can be improved by more than 60% by residual compression alone. The reported higher figures are presumably caused by additional factors, such as for example cold working*.

4. The present graphical procedure can also be used with some caution in the case of applied biaxial compression and tension. However it is not applicable to torsion which is a rather severe limitation of its usefulness. There is need for additional experimental work on the relief of residual stress by applied torsion before a suitable procedure can be suggested to account for the influence of residual stress on alternating torsion.

* It will be recalled that *RF* for notched specimens is the fatigue stress (limit) at the bottom of the notch. Because of stress gradient and grain size effect this value is generally larger than E.L., the endurance limit of notch free specimens (cf. Petersen, R. E., 1959, Notch Sensitivity, in: *Metal Fatigue*, Sines, G. and Waisman, J. L., ed., McGraw Hill Co., New York).

ON THE DUAL TYPE OF FRACTURE IN HARDENED CEMENT MORTARS

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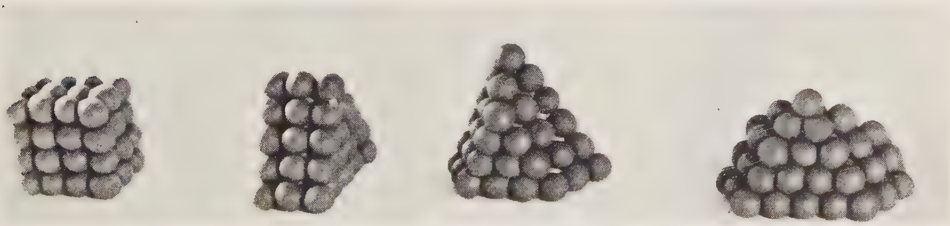
ABSTRACT

Experiments are described which show the similar dependence of such properties as bulk density, permeability to ultrasonic waves, compressive strength and mode of failure of cement mortars on sand content. The failure occurs either by seperation or by sliding.

Incident to these three categories of properties are two distinct forms of fracture, separated by an intermediate range of $c_v - 50\text{–}60\%$. This range includes the concentration corresponding to the geometrical volume ratio of equal spheres in a state of loose packing ($c_v=52.4\%$). Mortars below the intermediate range are brittle; those above it are plastic.

INTRODUCTION

Consider a system of equal rigid spheres dispersed in a non-rigid medium and brought into initial contact by gradual increase of concentration. There exists a transitional point, which may be termed the 'loose packing' stage, when the volume concentration of the system (defined as the volume ratio of the spheres to the total) is $c_v = 52.4\%$ (Figure 1). At higher concentrations the system is capable of transmitting stresses by friction and interlocking, a property absent at lower concentrations.



1 $c_v = 52.4\%$ Loose packing; 2 $c_v = 60.4\%$; 3 $c_v = 70\%$; 4 $c_v = 74\%$ Close packing

Figure 1
Four states of packing of equal spheres

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It would be logical to assume that systems with $c_v < 52.4\%$ would behave in a manner entirely different from those of $c_v > 52.4\%$, due to the introduction of friction in the latter. At the transitional point the continuous phase still exerts influence on the properties of the material, but on further concentration this influence diminishes until, at the ultimate point of $c_v = 74\%$ (called the 'close packing' stage — the theoretical maximum density possible*, Figure 1), friction and interlocking have become the predominant factors with the continuous phase converted into isolated pockets filling the voids. Thus, in the 52.4 — 74% range the system may be regarded as granular-like.

These elementary considerations are applicable to the study of the mechanical behaviour of cement mortar [1] which can as an approximation be regarded as a dispersion of rigid sand particles in a hardened cement paste [2]. Series of tests were conducted in order to confirm these assumptions experimentally.

1. EXPERIMENTAL DATA

Procedure

Mortars were made with Portland cement and standard Leighton Buzzard sand. Mix proportions varied from $c_v = 0$ to $c_v = 80\%$; the highest concentration practically possible. A water-cement ratio of 33% was used for all mixes. 96 cylindrical specimens, 4.0 cm in diameter, 9.7 cm in height, were cast in 16 sets of six samples each. Sand volume concentrations varied in 5% increments.

After mixing, casting and vibrating, the specimens were stored for 24 hours at 16°C and a relative humidity of 75–80%, demoulded and re-stored under the same condition,

At 28 days, the following tests were carried out:

a) *Bulk density.*

b) *Ultrasonic test.* The time of propagation of the ultrasonic waves through the cylinders was measured to 10^{-5} seconds by means of the Cawkell ultrasonic tester designed at the Road Research Laboratories, England, for quality control of concretes.

c) *Crushing test.* Immediately following, the specimens were crushed in a 30 ton Amsler hydraulic loading machine, under constant rate of stress, up to fracture. Both contact surfaces were overlaid with wax plates in order to eliminate lateral friction. Any changes observed in the specimens during loading, particularly prior to fracture, were noted. Im some cylinders, axial deformation was measured by means of a 10^{-4} inch dial gauge.

In all tests, measurement means were plotted against sand concentration.

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* In practice higher concentrations can be achieved because of the unavoidable presence of air voids in the system.

2. RESULTS

Dissimilarities between high and low c_v mortars were already observed at an early stage of preparation:

(i) At $c_v < 50\%$, considerable settlement of the free upper surface was observed during setting.

(ii) At $c_v > 50\%$, vibration resulted in segregation.

These two phenomena could be explained by the formation of a solid lattice around $c_v = 50\%$, which, once formed, prevents settling. Vibration, on the other hand, tends to destroy this lattice.

a) *Bulk Density*

The bulk density curve (Figure 2) is characterised by a moderate increase of density up to a maximum at $c_v = 50$–52%, and a steep decrease above it. The straight line in the diagram represents the theoretical function of bulk density based on the assumption of ideal non-porous mortar.

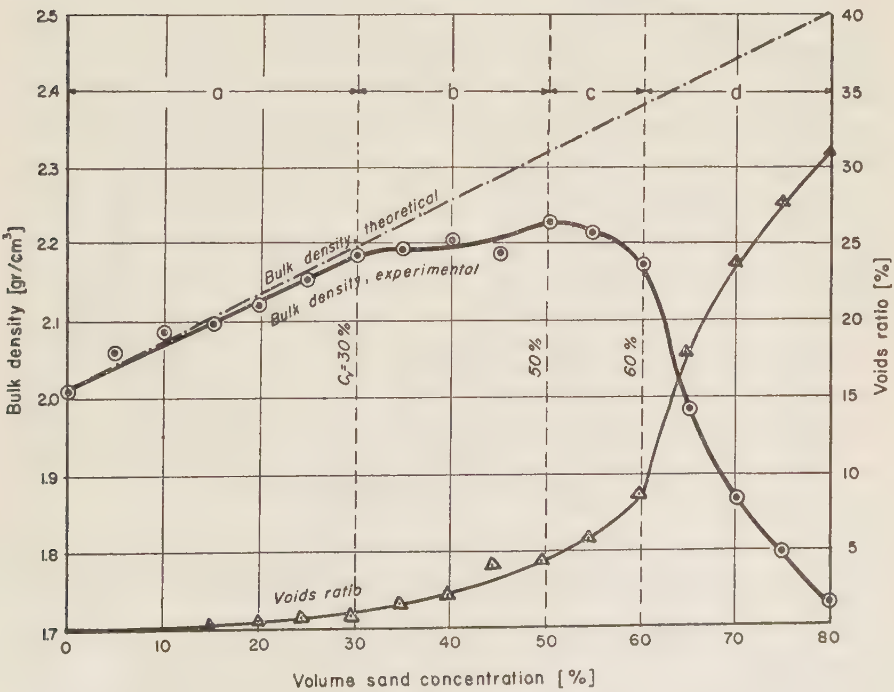


Figure 2

Bulk density and voids ratio vs. sand concentration

The voids ratio, calculated from the differences of theoretical and experimental density values, was also plotted on the same abscissa, showing three distinct phases:

1) An almost non-porous material up to $c_v = 30\%$.

2) A moderate increase in voids at $c_v = 30$–60%.

3) A steep rise above $c_v = 60\%$, up to an ultimate voids ratio of 32% at $c_v = 80\%$.

b) Ultrasonic test

The wave velocity graph (Figure 3) is very similar to the bulk density curve, with a slow increase up to a maximum at $c_v = 50$–54%, and a steep decrease above it.

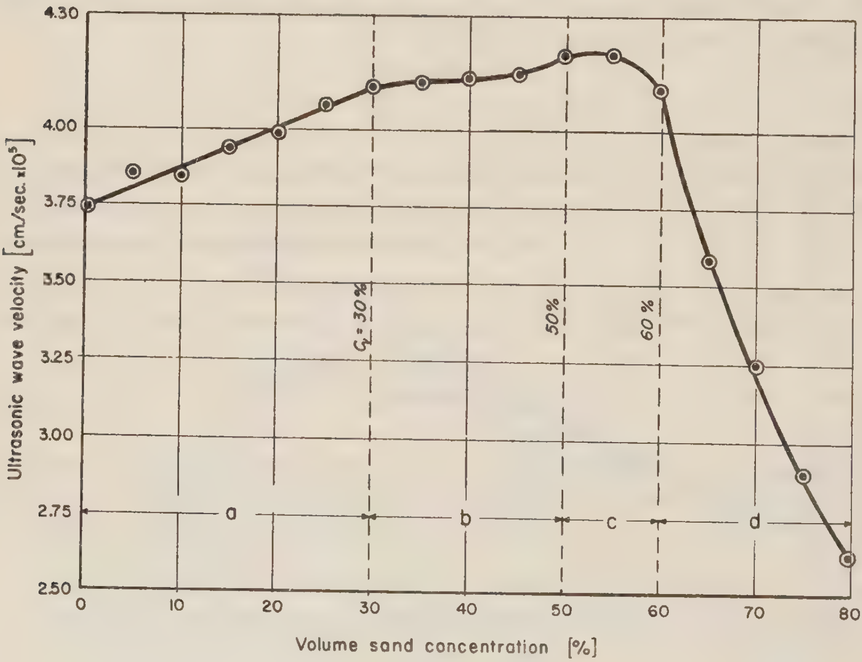


Figure 3

Ultrasonic wave velocity vs. sand concentration

c) Crushing test

The breaking-stress curve (Figure 4) shows high scatter at low c_v, and a maximum at $c_v = 30\%$. Thereafter it decreases, and at $c_v = 80\%$ attains a value of 10% of its

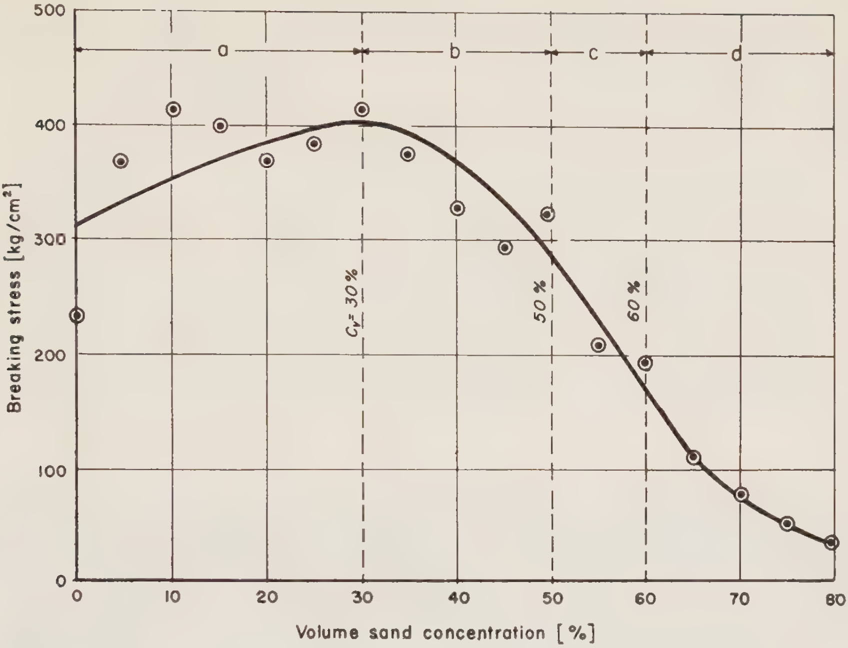


Figure 4

Breaking stress vs. sand concentration

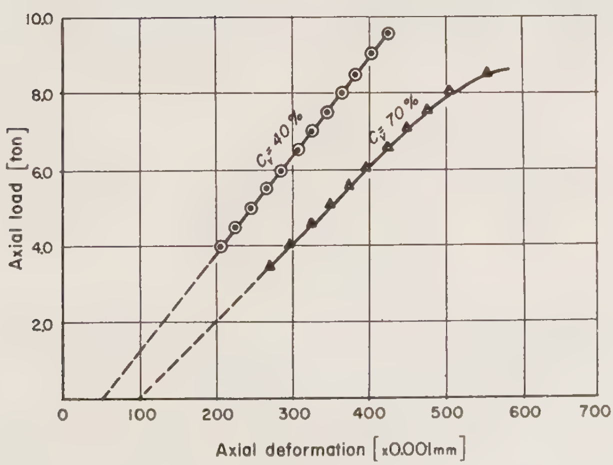


Figure 5

Two typical load-deformation curves

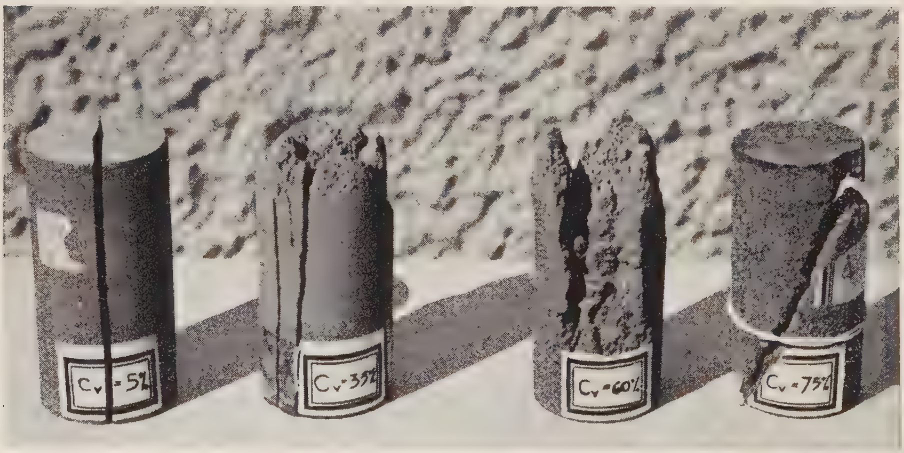
maximum. Figure 5 shows two typical load-deformation curves obtained for some specimens: linear throughout for low c_v, and with a deviation from linearity prior to fracture for high c_v.

3. DISCUSSION

Types of fracture

Compression tests showed two types of fracture, distinguishable also by behaviour prior to failure. Specimens of low c_v failed by separation, in the form of spontaneous splitting [3] parallel to the applied force, while those of low c_v failed by sliding, in the form of non-spontaneous diagonal shear [4], accompanied by crumbling*.

In addition, the separation range consists of two stages giving an overall picture as follows (Figure 6):



a b c d

Figure 6

Four typical forms of fracture obtained at crushing tests

a — separation into two parts
b — separation into several parts
c — combined separation and sliding.
d — pure sliding

1) Separation into two parts, observed for c_v below 30%. Failure is spontaneous and almost noiseless.

2) Separation into a larger number of parts, observed for the interval $c_v = 30–50\%$. Failure is again spontaneous, but in this case accompanied by a loud explosion.

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* This had been pointed out by R. l'Hermite in a private communication to Prof. M. Reiner.

3) Combined separation and sliding over an intermediate interval, $c_v = 50$–60%.

4) Pure sliding, observed for c_v above 60%. In this last case no damage was observed under the maximum load. In fact, failure could be avoided by reducing the load immediately upon reaching this maximum. On repeated loading, a lower maximum was reached without damage. Further repetition of the same procedure would eventually result in failure at a much lower load. This phenomenon can be attributed to the action of frictional forces slowing down the process of failure.

It is easily seen that the four stages in Figure 6 correspond to clearly-defined segments of the curves in Figures 2, 3, and 4, bounded by discontinuities.

In addition, Figures 2 and 3 show maxima at $c_v = 51$–53%, i.e. within the combined fracture range, and in close coincidence with loose packing state ($c_v = 52.4\%$).

It thus can be concluded that the division described above has physical significance.

4. CONCLUSIONS

The experiments show that in respect of mechanical properties cement mortar comprises two distinct materials depending on sand concentration, namely —

Brittle mortar (at $c_v < 50\%$)

Characterised by failure by separation and linear behaviour up to fracture; similar in all respects to hardened neat cement[3] and other brittle materials.

*Plastic mortar** (at $c_v > 60\%$)

Characterised by failure by sliding and deviation from linearity at about 60% of the breaking stress. It is safe to assume that this non-linearity is the result of internal shear fractures. This distinguishes its plasticity from that of metals where the large plastic deformation does not imply a fracture.

In a brittle mortar only one category of parameters — the elastic constants — is involved in the deformational behaviour, hence the linearity. In a plastic mortar, on the other hand, at least one more parameter is involved — that of friction, responsible for the non-linearity. In the intermediate stage ($50\% < c_v < 60\%$) the mortar combines the properties of both extremes.

ACKNOWLEDGMENT

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* The term plastic is used to characterise a deformation which is non-recoverable and takes place only after a 'yield point' has been exceeded. The first condition distinguishes 'plastic' from 'elastic'; the second distinguishes it from 'viscous'.

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LIMIT DESIGN OF A SYSTEM OF CROSSED BEAMS

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SUMMARY

Unlike elastic design, limit design is based only upon equilibrium conditions of the collapse mechanism.

Limit design by the simple plastic theory of crossed steel beams is presented for the case of proportional loading. Analysis includes "collapse lines" for the different mechanisms extended to the case of "active reactions".

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THERMAL BUCKLING OF SOLID WINGS*

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SUMMARY

The thermal buckling of solid wings with all edges free is analysed by the Rayleigh-Ritz Method. First wings of large aspect ratios are treated and the end effect is neglected. A double series of orthogonal polynomials is assumed for the deflection function and by the use of successively increasing numbers of terms the predominance of the simple torsional distortion is established for a number of typical temperature distributions for wings of constant thickness and of parabolic cross-section. The validity of the conclusion is verified for small aspect ratios.

Having been established as a good approximation the torsional analysis is modified to include also short wings. It is shown that the effects of the chordwise normal stresses and shear stresses cancel out, and that the critical temperature of a finite wing can be obtained from that of an infinite wing by multiplication by an "end effect factor". Calculations for a variety of typical cases show that this "end effect factor" is nearly unaffected by the type of temperature distribution or the shape of the cross section, but that the effect is significant for aspect ratios below 6 (more than 10%).

* Lecture presented at the Eighth Meeting of the Israel Union of Theoretical and Applied Mechanics, held in Haifa, April 9–10, 1958. An extended version of the lecture was later published in the *Journal of the Aero/space sciences*, **25**, No. 9, Sept. 1958.

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ON BOUNDARY CONDITIONS IN THE BENDING OF THIN ELASTIC PLATES

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SUMMARY

Using the classical (i.e. Kirchhoff's) theory of bending of thin elastic plates a third physical boundary condition can always be added to the two boundary conditions conventionally prescribed along each edge. This third boundary condition is derived from a perturbation of stress and strain which starts at the edge and vanishes at a short distance from it. This perturbation has no noticeable influence on the deflection of the plate. Taking into consideration this perturbation the contradiction inherent in the classical theory in connection with the boundary conditions can be eliminated, and a new physical interpretation for Kirchhoff's boundary condition is obtained.

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A LARGE DEFLECTION CRITERION FOR CIRCULAR PLATES

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SUMMARY

The theory of large deflections of symmetrically loaded circular plates leads to a pair of non-linear, second-order differential equations to which the Hilbert-Schmidt method of solving integral equations offers a complete solution. Based on the exact solution, a large-deflection criterion is derived, being more general than the conventional criterion of the deflection to thickness ratio; also taking into account the boundary conditions, dimensions of the plate, the elastic properties and the type of loading.

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SEMI-SPHERICAL HEAD GRINDING SET

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SUMMARY

Requirements: A mechanism for the grinding of a workpiece into an elongated semi-spherical head. The mechanism should be simple and automatic.

The requirement was met by mounting the workpiece on one end of a spinning shaft, free to precess around a point on its other end.

The workpiece precesses around a cone-shaped, inclined and revolving grinding wheel, pressing against it, and is thus ground into the required shape.

An oil damper is used to prevent infinite acceleration of precession.

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LECTURES PRESENTED AT THE EIGHTH CONFERENCE OF THE ISRAEL SOCIETY FOR THEORETICAL AND APPLIED MECHANICS, HELD IN HAIFA, APRIL 9-10, 1958

DEVIATION FROM PROPORTIONALITY IN THE LATTICE STRAIN–STRESS DIAGRAMS*

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SUMMARY

Deviations from proportionality in the lattice strain–stress diagram obtained from X-ray diffraction study of plastically deformed polycrystalline aggregates cannot be reconciled with Taylor's homogeneous five slip mechanism of plastic flow. To account for the observed discrepancies a heterogeneous slip mechanism is postulated. Part I describes the relevant theory and derives a lower bound for the expression of the applied stress causing yielding of a particular crystalline aggregate in uniaxial loading. Good qualitative agreement is obtained in Part II, when these expressions are compared with the results of an X-ray diffraction study of polycrystalline Aluminium alloy. Within the scope of this study it is concluded that the proposed theory is particularly suited to the analysis of plastic behaviour of surface grain aggregates at the initial stages of plastic deformation.

* Lecture presented at the Eighth Meeting of the Israel Society for Theoretical and Applied Mechanics, held in Haifa, April 9–10, 1958.

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THE MODULI OF AN ELASTIC SOLID CONTAINING SPHERICAL PARTICLES OF ANOTHER ELASTIC MATERIAL*

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SUMMARY

The elastic moduli of a homogeneous and isotropic material, containing spherical particles of another elastic material have been determined by the theory of elasticity.

It has been assumed that interaction between particles may be neglected and the solution is therefore restricted to the case of small volume concentration of particles.

The bulk and shear moduli have been obtained in the following form

$$\frac{\kappa^*}{\kappa_m} = 1 - 3\,(1 - \nu_m)\,\frac{1 - \kappa_p/\kappa_m}{2(1 - 2\nu_m) + (1 + \nu_m)\,\kappa_p/\kappa_m}\,c_v$$

$$\frac{\mu^*}{\mu_m} = 1 - 15\,(1 - \nu_m)\,\frac{1 - \mu_p/\mu_m}{7 - 5\nu_m + 2\,(4 - 5\nu_m)\mu_p/\mu_m}\,c_v$$

Here κ^* and μ^* are the moduli of the new material, κ_p and μ_p — those of the particles, κ_m and μ_m — those of the medium in which they are embedded and ν_m the Poisson's coefficient of the medium. c_v is the volume concentration of particles.

From these formulae known results for the case of rigid particles and spherical holes may be obtained as special cases.

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* Lecture presented at the Eighth Meeting of the Israel Society for Theoretical and Applied Mechanics, held in Haifa, April 9–10, 1958.

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יוצא לאור ע״י

מוסד ויצמן לפרסומים במדעי הטבע ובטכנולוגיה בישראל

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